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Some Elementary Theorems about Divisibility of 0-Cycles on Abelian Varieties Defined over Finite Fields

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1 Introduction

Let X be an abelian variety defined over a field k and let $L \in \mathrm{Pic}(X)$. Then the Riemann-Roch theorem implies that the degree of 0-cycle L^g is divisible by $g!$. An important example is when X is the product of an abelian variety with its dual $A \times A^\vee$, where A is of dimension n, thus $g = 2n$, and L is the Poincaré bundle, normalized so as to be trivial along the two 0-sections. In this case, the highest product L^g is just $g!$ times the origin, as a 0-cycle in $CH_0(A \times A^\vee)$. In this article we investigate the question of $g!$-divisibility of the 0-cycle L^g in $CH_0(X)$. We show that already for $g = 2$, there are counterexamples (see Remark 4.1). This answers negatively a question by B. Kahn (personal communications): there are no divided powers in the Chow groups of abelian varieties, not even in their étale motivic cohomology. Indeed, his question was the motivation to study divisibility of the 0-cycle L^g. Our method consists in relating this divisibility for $g = 2$ to the existence of theta characteristics on smooth projective curves over the given field. Over finite fields, Serre's theorem asserts that there are always theta characteristics. We derive from this that if $g = 2n$ and the field is finite, one always has 2-divisibility (see Theorem 2.1 and Remark 4.3). Thus, the 2-divisibility is an arithmetic statement (see Remark 4.2). In order to find a nontrivial class of invertible sheaves L for which one has $g!$-divisibility, one needs more arithmetic. In Theorem 3.1 we show that a principal polarization L of geometric origin has the strong property that L^g is $g!$-divisible as a 0-cycle. Aside of Serre's theorem mentioned above, the proof relies on (an adaption of) Mattuck's results

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on geometric polarization (see [4]), on abelian class field theory by Kato and Saito (see [3]), and on Bloch's theorem on 0-cycles on abelian varieties (see [2]). Perhaps it gives some hope that $g!$-divisibility is true in general over a finite field.

2 2-divisibility

Theorem 2.1. Let X be an abelian variety of dimension $g = 2n$ defined over a finite field. Let $L \in \mathrm{Pic}(X)$ be a line bundle. Then the 0-cycle L^g is divisible by 2 in the Chow group of 0-cycles of X. □

Proof. If A is a sufficiently ample line bundle, then $L \otimes A = B$ is very ample as well. One has $L^g \equiv A^g + B^g \bmod 2CH_0(X)$, thus one may assume that L is as ample as necessary. For L sufficiently ample, there is a finite field extension $k' \supset k$ of odd degree, such that the intersection $C \subset X \times_k k'$ of $(g-1)$ linear sections of $L \times_k k'$ in general position is smooth. One has $\omega_C = (L \times_k k')^{\otimes(g-1)}|_C \equiv (L \otimes_k k')|_C \bmod 2\,\mathrm{Pic}(C)$, thus, via the Gysin homomorphism $\mathrm{Pic}(C) \xrightarrow{\iota} CH_0(X \times_k k')$ one has $(L \times_k k')^g \equiv \iota_*(\omega_C) \bmod 2CH_0(X \times_k k')$. On the other hand, Serre's theorem [1, Remark 2, page 61] asserts that a smooth curve over a finite field admits a theta characteristic, that is, ω_C is 2-divisible in $\mathrm{Pic}(C)$. This shows that $(L \times_k k')^g \in 2CH_0(X \times_k k')$, thus by projection formula, since $k' \supset k$ is odd, then $L^g \in 2CH_0(X)$ as well. ■

3 $g!$-divisibility

Let C be a smooth projective curve of genus g defined over a finite field $k = \mathbb{F}_q$. Then C carries a 0-cycle p of degree 1. Let J be the Jacobian of C. The rest of this section would be trivial if $g \le 1$, thus we assume that $g \ge 2$. We consider the cycle map

$$\psi_p : C \longrightarrow J, \qquad y \longmapsto \mathcal{O}_C\big(y - \deg(y) \cdot p\big). \tag{3.1}$$

This cycle map induces a birational morphism which we still denote by ψ_p,

$$\psi_p : \mathrm{Sym}^g(C) \longrightarrow J, \qquad \big(x_1, \dots, x_g\big) \longmapsto \otimes_{i=1}^g \psi_p\big(x_i\big). \tag{3.2}$$

In particular, writing $p = \sum_i m_i p_i = q_1 - q_2$, with q_i effective, one has $\deg(q_1) - \deg(q_2) = \sum_i m_i \deg(p_i) = 1$. We denote by $\pi : C^g \to \mathrm{Sym}^g(C)$ the quotient map. It defines the divisor

$$D_p = \sum_i m_i \big(\psi_p \circ \pi\big)_* \big(C^{g-1} \times p_i\big), \qquad L_p = \mathcal{O}_J\big(D_p\big) \in \mathrm{Pic}(J). \tag{3.3}$$

We know that L_p is a principal polarization, thus the 0-cycle L_p^g has degree $g!$. The purpose of this section is to show the following theorem.

Theorem 3.1. The 0-cycle L_p^g in the Chow group $CH_0(J)$ of 0-cycles of J is divisible by $g!$ in $CH_0(J)$, that is, there is a 0-cycle $\xi \in CH_0(J)$ of degree 1 with $L_p^g = g! \cdot \xi \in CH_0(J)$. $\square$

Proof. Since the 0-cycle p of degree 1 will not change during the proof, we simplify the notation and set $\psi = \psi_p$, $D = D_p$, and $L = L_p$. We consider the Poincaré bundle $P = p_1^*L \otimes p_2^*L \otimes \mu^*L^{-1} \in \mathrm{Pic}(J \times J)$, where $\mu : J \times J \to J$, $\mu(x, y) = x + y$. Via the cycle map

$$\iota_* : \mathrm{Alb}(C) = \mathrm{Pic}(C)^0 = \mathrm{Pic}(J)^0 \xrightarrow{\psi_*} CH_0(J)^{\deg = 0} \xrightarrow{h} J(k), \tag{3.4}$$

where h is the Albanese mapping of J, one defines

$$\iota_* \omega_C\big(-2(g-1)p\big) = y \in J(k). \tag{3.5}$$

We now adapt [4, Section 6] to the situation where p is not necessarily a k-rational point of C, but only a 0-cycle of degree 1. One defines the involution

$$\delta : J \longrightarrow J, \qquad x \longmapsto -x + y = \tau_y \circ (-1)^*(x) = (-1)^* \circ \tau_{-y}, \tag{3.6}$$

where τ_y is the translation by y while (-1) is the multiplication by -1. We set

$$\ell = \psi^* L, \qquad \mathcal{P} = (\psi \times 1)^* P = p_1^* \ell \otimes p_2^* L \otimes (\psi \times 1)^* \mu^* L^{-1}. \tag{3.7}$$

As in [5, page 249], for $d > 2g - 2$, the Riemann-Roch theorem asserts that

$$E_d := p_{2*}\big(\mathcal{P} \otimes p_1^* \mathcal{O}_C(dp)\big) = Rp_{2*}\big(\mathcal{P} \otimes p_1^* \mathcal{O}_C(dp)\big) \tag{3.8}$$

is a vector bundle. One has

$$\otimes_i \mathcal{P}|_{p_i \times J}^{\otimes m_i} = \mathcal{P}|_{C \times \{0\}} = \mathcal{O}_J. \tag{3.9}$$

This implies that for any natural number $f > 0$, one has

$$\mathcal{P}_{fq_1 \times J} \otimes \mathcal{P}_{fq_2 \times J}^{-1} = \mathcal{P}|_{C \times \{0\}} = \mathcal{O}_J. \tag{3.10}$$

For two natural numbers $e > f > 0$, one has the diagram

$$\begin{tikzcd} \mathcal{P} \otimes p_1^* \mathcal{O}_C\big((e-f)q_1 - eq_2\big) \arrow[r, "fq_1"] \arrow[d, "fq_2"'] & \mathcal{P} \otimes p_1^* \mathcal{O}_C(ep) \\ \mathcal{P} \otimes p_1^* \mathcal{O}_C\big((e-f)p\big), & \end{tikzcd} \tag{3.11}$$

where the horizontal map is induced by $\mathcal{O}_C \to \mathcal{O}_C(fq_1)$, and the vertical one by $\mathcal{O}_C \to \mathcal{O}_C(fq_2)$. Thus one has the following relation in $K_0(J)$:

$$\begin{aligned} &Rp_{2*}\big(\mathcal{P} \otimes p_1^*\mathcal{O}_C(ep)\big) - Rp_{2*}\big(\mathcal{P} \otimes p_1^*\mathcal{O}_C\big((e-f)p\big)\big) \\ &\quad = p_{2*}\Big(\mathcal{P} \otimes p_1^*\mathcal{O}_C\big((e-f)p\big)\big|_{fq_2\times J}\Big)^{-1} \otimes p_{2*}\Big(\mathcal{P} \otimes p_1^*\mathcal{O}_C(ep)\big|_{fq_1\times J}\Big) \\ &\quad = 0 \quad (\text{by } (3.10)). \end{aligned} \tag{3.12}$$

For $d > 2g-2$, we then have in the Grothendieck group $K_0(J)$,

$$p_{2*}\big(\mathcal{P} \otimes p_1^*\mathcal{O}_C(dp)\big) + R^1p_{2*}\big(\mathcal{P} \otimes \mathcal{O}_C\big((2g-2-d)p\big)\big) = 0. \tag{3.13}$$

On the other hand, one has

$$\delta^*E_d = p_{2*}\big(\mathcal{P}^{-1} \otimes p_1^*\omega_C\big((d-2g+2)p\big)\big), \tag{3.14}$$

and Serre duality implies that

$$\delta^*\big(E_d^\vee\big) = R^1p_{2*}\big(\mathcal{P} \otimes p_1^*\big(\mathcal{O}\big((2g-2-d)p\big)\big)\big) \tag{3.15}$$

(see [5, page 249]). The involution δ acts on J, thus on the Chow groups of J. Thus (3.13) and (3.15) imply that

$$\delta^*c_1\big(E_d\big) = c_1\big(E_d\big) =: M, \qquad M^2 = \delta^*M^2 = c_2\big(E_d\big) + \delta^*c_2\big(E_d\big). \tag{3.16}$$

Thus we have

$$M^g = M^2 \cdot M^{g-2} = M^{g-2}c_2\big(E_d\big) + \delta^*M^{g-2} \cdot \delta^*c_2\big(E_d\big) = Z + \delta^*Z \tag{3.17}$$

for

$$Z := c_2\big(E_d\big) \cdot M^{g-2} \in CH_0(J), \quad \deg Z = n, \quad 2n = \deg M^g. \tag{3.18}$$

We set

$$Z' = Z - n\{0\} \in CH_0(J)^{\deg=0}. \tag{3.19}$$

By abelian class field theory for 0-cycles on varieties defined over finite fields, see [3, Corollary, page 274], the Albanese mapping h is an isomorphism. This allows to identify explicitly δ^* on $CH_0(J)$. Indeed, if $W \in CH_0(J)$ has degree n, and if $z \in J(k)$, then

$$\begin{aligned} h\big(\tau_z^*(W) - n\{0\}\big) &= \tau_z^*h\big(W - n\{0\}\big) + h\big(\tau_z^*n\{0\} - n\{0\}\big) \\ &= \tau_z^*h\big(W - n\{0\}\big) + nz. \end{aligned} \tag{3.20}$$

On the other hand, Bloch's theorem [2, Theorem 3.1] asserts that the second Pontryagin product dies in $CH_0(J\times_k\bar{k})$, where $\bar{k}$ is the algebraic closure of k, thus by class field theory again [3, Proposition 9, page 274], it dies in $CH_0(X)$. Thus

$$\tau_z^* a = a \quad \forall a \in CH_0(X)^{\deg=0}. \tag{3.21}$$

Thus (3.20) and (3.21) imply that

$$h\big(\tau_z^*(W) - n\{0\}\big) = h\big(W - n\{0\}\big) + nz. \tag{3.22}$$

On the other hand,

$$h\big((-1)^* W - n\{0\}\big) = -h\big(W - n\{0\}\big). \tag{3.23}$$

Thus, (3.17), (3.22), and (3.23) imply that

$$h\big(M^g - 2n\{0\}\big) = -ny. \tag{3.24}$$

We now apply again Serre's theorem [1, Remark 2, page 61], which asserts that over a finite field, a smooth projective curve has a theta divisor. Thus, $\omega_C(-(2g-2)p) \in 2\,\mathrm{Pic}^0(C)$ and a fortiori, via the Gysin homomorphism ψ_*, one has

$$y = 2\xi_0 \in CH_0(J)^{\deg=0} \quad \text{for some } \xi_0 \in CH_0(J)^{\deg=0}. \tag{3.25}$$

Thus (3.24), (3.25), and [3] imply that

$$M^g = 2n\big(\{0\} - \xi_0\big). \tag{3.26}$$

It remains to compare $M^\vee$ and L. As a divisor, one has $M^\vee \otimes_k \bar{k} = \{\mathcal{L} \in \mathrm{Pic}^0(J)(\bar{k}), \Gamma(C \times_k \bar{k}, \mathcal{L}((g-1)p)) \neq 0\}$. Thus $M^\vee \otimes_k \bar{k} = \mathcal{O}_{J\times_k \bar{k}}(D)$, as we know, both underlying divisors are physically the same and they are both reduced. Thus $M^\vee = \mathcal{O}_J(D) \otimes \mathcal{L}$ for some $\mathcal{L} \in \mathrm{Pic}^0(J)$ which is torsion. On the other hand, the map $J \to \mathrm{Pic}^0(J)$, $a \mapsto \tau_a^* \mathcal{O}_J(D) \otimes \mathcal{O}_J(D)^{-1}$ is an isomorphism over $\bar{k}$, thus it is an isomorphism over k. This implies that $M^\vee = \tau_a^* \mathcal{O}_J(D)$ for some $a \in J$. Thus (3.26) implies that

$$L^g = 2n\tau_a^*\big(\xi_0 - \{0\}\big), \quad 2n = g!. \tag{3.27}$$

This finishes the proof. ■

4 Remarks

Remark 4.1. B. Kahn asked whether étale motivic cohomology of abelian varieties has divided powers. Theorem 2.1, read backwards, yields a negative answer. Let k be a field and let C be a genus 2 curve defined over this field with the two properties that it carries a 0-cycle p of degree 1 and it does not have a theta characteristic. Let J be the Jacobian of C and let $\psi_p =: \psi : C \to J$ be the cycle map assigned to the choice of p. Since the composite map $\iota_* : \mathrm{Alb}(C) = \mathrm{Pic}(C)^0 = \mathrm{Pic}(J)^0 \xrightarrow{\psi_*} CH_0(J)^{\deg=0} \xrightarrow{h} J(k)$ is an isomorphism, one has that $\iota_*\omega_C(2(g-1)p) \in J(k)$ is not 2-divisible. Thus, a fortiori, the class of the Gysin image of ω_C will not be 2-divisible in any cohomology which has the property that it maps to étale cohomology and the kernel maps to the Albanese; for example, étale motivic cohomology.

It remains to give a concrete curve. One could take for k the function field of the fine moduli space of pointed genus 2 curves with some level over a given algebraically closed field F. This has transcendence degree 3 over F and is of course very large. Here is an example due to J.-P. Serre over the field $k = \mathbb{C}(t)$ of cohomological dimension 1: C is defined by its hyperelliptic equation $y^2 = x^6 - x - t$. It has two rational points at ∞. The Galois group of $x^6 - x - t$ is the symmetric group in six letters, which acts with two orbits on the space of theta characteristics over $\bar{k}$, one with six elements and the other with ten.

Remark 4.2. This remark arose in discussions with E. Viehweg in view of Theorem 2.1. If X is a product of curves $X = C_1 \times \cdots \times C_g$ over a field k, then the Pic functor is quadratic after Mumford, which means that a line bundle L on X is a sum of line bundles L_{ij} coming via pullback from only two factors (ij), $i \neq j$. Thus the expansion of L^g will have two kinds of summands. First it will have those of type $L_{i_1j_1} \cdots L_{i_gj_g}$ with all pairs (i_c, j_c) being different. The coefficient of such a summand is $g!$, thus this term is $g!$-divisible. Then it will have those of type $L^2_{i_1j_1} \cdots L^2_{i_aj_a} \cdot L_{i_{a+1}j_{a+1}} \cdots L_{i_{a+b}j_{a+b}}$ with all pairs (i_c, j_c) being different and $2a + b = g$. The coefficient of such a summand is $g!/2^a$. Thus $g!$-divisibility of any L^g on a product of g curves splits into two kinds of divisibility. Over any field k, geometry always forces $(g!/2^a)$-divisibility for $a = [g/2]$, where $[c]$ means the integral part of a real number c. On the other hand, let k be a field which has the property that any curve has a theta characteristic (e.g., a finite field (see [1, Remark 2, page 61])). Then the argument of Theorem 2.1 implies that if $L \in \mathrm{Pic}(C_1 \times C_2)$, then L^2 is 2-divisible. Indeed, one reduces as in the proof of Theorem 2.1 to the case where $L = \mathcal{O}(\Gamma)$ for a smooth curve $\Gamma \subset C_1 \times C_2$, and by the adjunction formula one has $L^2 = i_{\Gamma *}\omega_\Gamma - p_1^*\omega_{C_1} \cdot \Gamma - p_2^*\omega_{C_2} \cdot \Gamma$, where $p_i : C_1 \times C_2 \to C_i$ are the projections and $i_\Gamma : \Gamma \hookrightarrow C_1 \times C_2$ is the closed embedding. Thus over such a field, L^g is always $g!$-divisible in $CH_0(C_1 \times \cdots \times C_g)$.

Remark 4.3. The conclusion of Theorem 2.1 is of course true over any field k over which any smooth projective curve has a theta characteristic.

Acknowledgments

The article is an answer to a question by B. Kahn. S. Bloch explained to us long ago the use of abelian class field theory to compute 0-cycles over finite field. This article relies on this idea. J.-P. Serre explained to us his theta theorem. We thank them all, and also E. Howe, A. Polishchuk, and E. Viehweg for answering our questions. In particular, Remark 4.2 comes from a discussion with E. Viehweg.

References

[1] M. F. Atiyah, *Riemann surfaces and spin structures*, Ann. Sci. École Norm. Sup. (4) **4** (1971), 47–62.

[2] S. Bloch, *Some elementary theorems about algebraic cycles on Abelian varieties*, Invent. Math. **37** (1976), no. 3, 215–228.

[3] K. Kato and S. Saito, *Unramified class field theory of arithmetical surfaces*, Ann. of Math. (2) **118** (1983), no. 2, 241–275.

[4] A. Mattuck, *Symmetric products and Jacobians*, Amer. J. Math. **83** (1961), 189–206.

[5] A. Polishchuk, *Abelian Varieties*, *Theta Functions and the Fourier Transform*, Cambridge Tracts in Mathematics, vol. 153, Cambridge University Press, Cambridge, 2003.

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